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CFT hydrodynamics: symmetries, exact solutions and gravity

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ABSTRACT: We consider the hydrodynamics of relativistic conformal field theories at finite temperature and its slow motions limit, where it reduces to the incompressible Navier-Stokes equations. The symmetries of the equations and their solutions are analyzed. We construct exact solutions with finite time singularities of one-dimensional relativistic conformal hydrodynamic motions, and use them to generate multi-dimensional solutions via special conformal transformations. These solutions, however, are shown to have no non-trivial slow motions limit. A simple non-equilibrium steady state in the form of a shock solution is constructed, and its inner structure is analyzed. We demonstrate that the derivation of the gravitational dual description of conformal hydrodynamics is analogous to the derivation of hydrodynamics equations from the Boltzmann equation. The shock solution is shown to correspond to a domain-wall solution in gravity. We show that the solutions to the non-relativistic incompressible Navier-Stokes equations play a special role in the construction of global solutions to gravity.

KEYWORDS: Gauge-gravity correspondence, AdS-CFT Correspondence, Classical Theories of Gravity

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Contents

1	Introduction	1
2	Symmetries of CFT hydrodynamics	5
2.1	Special conformal symmetries of relativistic motions	6
2.2	Special conformal symmetries of non-relativistic motions	7
3	Exact solutions of ideal CFT hydrodynamics	8
3.1	One-dimensional motions and finite time singularities in ideal CFT hydrodynamics	8
3.2	Special conformal symmetry generation of higher-dimensional solutions	11
3.3	The non-relativistic limit	11
4	Shock solutions	12
4.1	Shocks-global solutions in ideal hydrodynamics	12
4.2	Special conformal transformation of shocks	14
4.3	Shocks in viscous CFT hydrodynamics	15
4.4	The viscous shock solution at a small Knudsen number	18
4.5	The shock solution at a Knudsen number of order unity	19
4.6	How is it possible to have $Kn \ll 1$, while the viscous term still being important?	20
5	Gravity and fluid dynamics	21
5.1	The importance of the solutions to non-relativistic hydrodynamics for the construction of global solutions to gravity	23
5.2	Global solution for gravity from a small Knudsen number viscous shock	23

1 Introduction

The hydrodynamics of relativistic conformal field theories has attracted much attention recently, largely in view of the AdS/CFT correspondence between gravitational theories on asymptotically Anti-de-Sitter (AdS) spaces and CFTs [1] (for a review see [2]). Hydrodynamics gives a universal description of the large-time dynamics of the theory: starting from an arbitrary initial state of the CFT, within the correlation time τ_{cor} the system approaches the state of local thermal equilibrium. At $t \gg \tau_{cor}$ the evolution is mainly the evolution of the parameters of the local equilibrium described by hydrodynamics. On the basis of the AdS/CFT correspondence one then expects that the large-time dynamics of gravity can be obtained as a dual description of the CFT hydrodynamics [3, 4].

A complete (compressible) hydrodynamics is described by five fields: the three velocity components, the temperature and the particle density [5–7]. In a CFT there is no locally conserved charge corresponding to the particle density [6, 8]. As a result, conformal hydrodynamics is described by only four fields: the three velocity components and the temperature.

Hydrodynamics applies under the condition that the correlation length of the fluid l_{cor} is much smaller than the characteristic scale L of variations of the macroscopic fields. In order to characterize this, one introduces the dimensionless Knudsen number

$$Kn \equiv l_{cor}/L . \tag{1.1}$$

Since the only dimensionfull parameter is the characteristic temperature of the fluid T , one has by dimensional analysis,

$$l_{cor} = (\hbar c/k_B T)G(\lambda) , \tag{1.2}$$

where λ denotes all the dimensionless parameters of the CFT. The function $G(\lambda)$ characterizes the CFT.

The stress-energy tensor of the CFT obeys

$$\partial_\nu T^{\mu\nu} = 0, \quad T^\mu_\mu = 0 . \tag{1.3}$$

The equations of relativistic hydrodynamics are determined by the constitutive relation expressing $T^{\mu\nu}$ in terms of the temperature $T(x)$ and the four-velocity field $u^\mu(x)$ satisfying $u_\mu u^\mu = -1$. Here u^μ and T determine the local thermal equilibrium distribution of the fluid. The constitutive relation has the form of a series in the small parameter $Kn \ll 1$,

$$T^{\mu\nu}(x) = \sum_{l=0}^{\infty} T_l^{\mu\nu}(x), \quad T_l^{\mu\nu} \sim (Kn)^l, \tag{1.4}$$

where $T_l^{\mu\nu}(x)$ is determined by the local values of u^μ and T and their derivatives of a finite order. The smallness of $T_l^{\mu\nu}$ arises because it involves either the l -th derivative of u^μ or T or because it contains the corresponding power of a lower-order derivative. Keeping only the first term in the series gives ideal hydrodynamics, within which the entropy is conserved and the entropy density per unit volume σ obeys a conservation law $\partial_\mu(\sigma u^\mu) = 0$. Dissipative hydrodynamics arises when one keeps the first two terms in the series.

The ideal hydrodynamics approximation for $T^{\mu\nu}$ does not contain the spatial derivatives of the fields. The stress-energy tensor reads (up to a multiplicative constant)

$$T_{\mu\nu} = T^4 [\eta_{\mu\nu} + 4u_\mu u_\nu] , \tag{1.5}$$

where $\eta_{\mu\nu} = \text{diag}[-, +, +, +]$.

The dissipative hydrodynamics is obtained by keeping the $l = 1$ term in the series in eq. (1.4). In the Landau frame [6, 9] the stress-energy tensor reads (up to a multiplicative constant)

$$T_{\mu\nu} = T^4 [\eta_{\mu\nu} + 4u_\mu u_\nu] - c\eta\sigma_{\mu\nu} , \tag{1.6}$$

where $\sigma_{\mu\nu}$ obeys $\sigma_{\mu\nu}u^\nu = 0$ and is given by

$$\sigma_{\mu\nu} = (\partial_\mu u_\nu + \partial_\nu u_\mu + u_\nu u^\rho \partial_\rho u_\mu + u_\mu u^\rho \partial_\rho u_\nu) - \frac{2}{3} \partial_\alpha u^\alpha [\eta_{\mu\nu} + u_\mu u_\nu] . \quad (1.7)$$

The dissipative hydrodynamics of a CFT is determined by only one kinetic coefficient - the shear viscosity η . The bulk viscosity ζ vanishes for the CFT, while the absence of the particle number conservation and the use of the Landau frame allow to avoid the use of heat conductivity, which is not an independent coefficient here [8]. Dimensional analysis dictates that $\eta = F(\lambda)T^3$, where $F(\lambda)$ is a function characterizing the CFT and we again omit a multiplicative constant in $T^{\mu\nu}$.

The hydrodynamics of relativistic conformal field theories is intrinsically relativistic as is the microscopic dynamics. However, it has been shown in [10], that the limit of non-relativistic macroscopic motions of a CFT hydrodynamics leads to the non-relativistic incompressible Euler and Navier-Stokes equations for ideal and dissipative hydrodynamics of the CFT, respectively (see also [11]). For ideal hydrodynamics the implication follows by noting that the equations (1.3) written in terms of \mathbf{v} defined by $u^\mu = (\gamma, \gamma\mathbf{v}/c)$ and a variable P equal to $c^2 \ln T$ up to an additive constant, take the form

$$\frac{1}{c^2} \left[\frac{\partial P}{\partial t} + \frac{2c^2}{3c^2 - v^2} (\mathbf{v} \cdot \nabla) P \right] = -\frac{c^2}{3c^2 - v^2} \nabla \cdot \mathbf{v}, \quad (1.8)$$

$$\frac{\partial v_i}{\partial t} + (\mathbf{v} \cdot \nabla) v_i = - \left(1 - \frac{v^2}{c^2} \right) \left[\delta_{ij} - \frac{2v_i v_j}{3c^2 - v^2} \right] \nabla_j P + \frac{(c^2 - v^2) v_i (\nabla \cdot \mathbf{v})}{3c^2 - v^2} . \quad (1.9)$$

Now if we consider the solutions for which \mathbf{v} remains finite in the limit $c \rightarrow \infty$, then the limiting field \mathbf{v} obeys

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \frac{\nabla \cdot \mathbf{v}}{3} \mathbf{v}, \quad -\frac{1}{3} \nabla \cdot \mathbf{v} = \frac{1}{c^2} \frac{\partial P}{\partial t} . \quad (1.10)$$

The first equation above implies that ∇P must be finite at $c \rightarrow \infty$. However it is still possible that P contains a function of time leading to a finite spatially constant divergence of velocity. The general form of non-relativistic dynamics is therefore

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P - a(t) \mathbf{v}, \quad \nabla \cdot \mathbf{v} = -3a(t) . \quad (1.11)$$

If we now impose the condition that \mathbf{v} remains finite at large distances (which is valid in most physical situations), then we must require that $a(t) \equiv 0$. We conclude that non-relativistic, finite at infinity, motions of the CFT obey the incompressible Euler equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P, \quad \nabla \cdot \mathbf{v} = 0 . \quad (1.12)$$

Analogous considerations hold for the viscous hydrodynamics where the limiting field \mathbf{v} obeys the incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0 . \quad (1.13)$$

The kinematic viscosity ν is given by

$$\nu = \hbar c^2 F(\lambda) / 4k_B T_0 . \tag{1.14}$$

T_0 is the main component of the temperature, which is approximately constant in the considered limit,

$$T = T_0 \left[1 + \frac{P}{c^2} + O\left(\frac{1}{c}\right) \right] , \tag{1.15}$$

see [10]. Thus, a relativistic conformal field theory contains the incompressible Euler and Navier-Stokes equations inside it.

The solutions to the non-relativistic incompressible Navier-Stokes equations have special importance in light of the AdS/CFT correspondence: they allow to construct an explicit approximate solutions to the five-dimensional Einstein equations with negative cosmological constant

$$R_{mn} + 4g_{mn} = 0, \quad R = -20, \tag{1.16}$$

where we use $R_{AdS} = 1$. If \mathbf{v} and P solve eqs. (1.13) and obey the condition $Kn \ll 1$, then the metric g_0 defined by ($\hbar = k_B = 1$)

$$(g_0)_{mn} dy^m dy^n = -2u_\mu(x^\alpha) dx^\mu dr + \pi^4 T^4(x^\alpha) r^{-2} u_\mu(x^\alpha) u_\nu(x^\alpha) dx^\mu dx^\nu + r^2 \eta_{\mu\nu} dx^\mu dx^\nu, \tag{1.17}$$

$$y = (x^\mu, r), \quad u^\mu = \left(\frac{1}{\sqrt{1 - v^2/c^2}}, \frac{\mathbf{v}/c}{\sqrt{1 - v^2/c^2}} \right), \quad T = \frac{c^2}{4\pi\nu} + \frac{P}{4\pi\nu}, \tag{1.18}$$

has a value of $R_{mn} + 4g_{mn}$, which is small both in Kn and v/c (see the details in section 5).

Consider a typical solution of eq. (1.13), which is turbulent (the majority of flows in nature are turbulent). There are two important scales in the solution: the outer scale of turbulence L_O , dictated by the boundary or initial conditions, and the Kolmogorov scale

$$l_K \sim (\nu^3/\epsilon)^{1/4} \ll L_O, \tag{1.19}$$

where ϵ is the energy dissipation per unit volume [5]. At the scale l_K the flow becomes smooth (here we make a rough estimate neglecting intermittency), so the Knudsen number obeys $Kn \sim l_{cor}/l_K$. Using $\epsilon \sim v_c^3/L_O$ where v_c is characteristic value of the fluctuating component of the flow velocity, one may reexpress Kn in terms of L_O and v_c as

$$Kn \sim (l_{cor}/L_O)^{1/4} (v_c/c)^{3/4} . \tag{1.20}$$

Here we used the fact that generally $F(\lambda) \sim G(\lambda)$, see [10, 12]. One sees that one can realize $Kn \ll 1$ by dialing the outer scale of turbulence L_O , and the characteristic fluctuating component of the velocity v_c . The possibility of having a globally defined hydrodynamic flow with a uniformly small Knudsen number is specific for non-relativistic flows, and it normally does not exist for relativistic (compressible) flows, see details in sections 4 and 5.

In this paper we consider the hydrodynamics of relativistic conformal field theories at finite temperature and its slow motions limit, where it reduces to the incompressible Navier-Stokes equations. The symmetries of the equations and their solutions are analyzed. We construct exact solutions with finite time singularities of one-dimensional relativistic conformal hydrodynamic motions, and use them to generate multi-dimensional solutions via special conformal transformations. These solutions, however, are shown to have no non-trivial slow motions limit. A simple non-equilibrium steady state in the form of a shock solution is constructed, and its inner structure is analyzed. We demonstrate that the derivation of the gravitational dual description of conformal hydrodynamics is analogous to the derivation of hydrodynamics equations from the Boltzmann equation. The shock solution is shown to correspond to a domain-wall solution in gravity. We show that the solutions to the non-relativistic incompressible Navier-Stokes equations play a special role in the construction of global solutions to gravity.

The paper is organized as follows. In the section 2 we consider the symmetries of relativistic conformal hydrodynamics and their relation to the symmetries of the incompressible Euler and Navier-Stokes equations in the limit of slow motions. In section 3 we construct exact solutions of ideal CFT hydrodynamics. We analyse one-dimensional macroscopic motions, introduce Riemann variables and provide a simple wave solution that becomes singular in a finite time. We use the solution in order to generate multi-dimensional solutions via special conformal transformations. We show, however, that these solutions have no non-trivial slow motions limit. In section 4 we discuss weak solutions of ideal hydrodynamics defined at all times. We construct a stationary shock solution as an example of a global weak solution. We describe its inner structure due to viscosity, its conformal transformation and its non-relativistic limit. In section 5 we argue that the derivation of the gravitational dual description of conformal hydrodynamics is analogous to the derivation of hydrodynamics equations from the Boltzmann equation. We show that the shock solution corresponds to a domain-wall solution in gravity.

2 Symmetries of CFT hydrodynamics

Here we consider some special properties of the CFT hydrodynamics related to the underlying symmetries of the CFT, extending some aspects of the analysis of [11]. The conformal group is the symmetry group of the CFT equations (1.3), and it maps solutions into solutions. The group includes the Poincare group, a dilatation and four special conformal transformations. The Poincare group reduces in the limit of slow motions to the Galilean group, including Galilean transformations and space and time translations, which is a symmetry group of the Navier-Stokes equations.

Consider next the dilatation that acts as:

$$x^\mu \rightarrow \lambda x^\mu, \quad u^\mu(x^\alpha) \rightarrow u^\mu(\lambda x^\alpha), \quad T(x^\alpha) \rightarrow \lambda^{-1} T(\lambda x^\alpha). \quad (2.1)$$

The dilatation is a symmetry of the CFT equations (1.3), and it remains a symmetry for ideal hydrodynamics in the limit of non-relativistic macroscopic motions, i.e. a symmetry

of the Euler equations. It acts as

$$t \rightarrow \lambda t, \quad x^i \rightarrow \lambda x^i, \quad v^i \rightarrow v^i. \quad (2.2)$$

For viscous hydrodynamics in the limit of non-relativistic macroscopic motions, one has the dimensionfull kinematic viscosity that breaks this symmetry, and indeed, (2.2) is not a symmetry of the Navier-Stokes equations.

The Euler equations have a bigger symmetry [5], which is generated by (2.2) together with

$$t \rightarrow \mu t, \quad x^i \rightarrow x^i, \quad v^i \rightarrow \mu^{-1} v^i. \quad (2.3)$$

However, the symmetry (2.3) does not follow from the symmetry group of relativistic conformal hydrodynamics in the limit of slow motions, cf. below. A combination of the two symmetries (2.2) and (2.3) with the same transformation parameter $\lambda = \mu = \alpha$ yields

$$t \rightarrow \alpha^2 t, \quad x^i \rightarrow \alpha x^i, \quad v^i \rightarrow \alpha^{-1} v^i, \quad (2.4)$$

which is a symmetry of the non-relativistic viscous equations. In view of the above discussion, we conclude that also this symmetry does not follow from the symmetry group of relativistic conformal hydrodynamics in the limit of slow motions.

2.1 Special conformal symmetries of relativistic motions

The special conformal transformations of the space-time coordinates x^μ are given by

$$\Phi^\mu(x, b) = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}, \quad (2.5)$$

where b^μ is a constant four-vector. They satisfy the identity

$$\eta^{\mu\nu} (\partial_\mu \Phi^\alpha) (\partial_\nu \Phi^\beta) = \frac{\eta^{\alpha\beta}}{(1 + 2b \cdot x + b^2 x^2)^2}. \quad (2.6)$$

A main distinguishing property of the CFT hydrodynamics is that the special conformal transformations allow to generate new solutions out of a given one. Consider a stress-energy tensor $T^{\mu\nu}(x)$ of the conformal field theory, i.e. that satisfies equations (1.3). Then, special conformal transformations produce a four-parametric family

$$T_{\mu\nu}(x, b) = \frac{(\partial_\mu \Phi^\alpha) (\partial_\nu \Phi^\beta)}{(1 + 2b \cdot x + b^2 x^2)^2} T_{\alpha\beta} [\Phi^\rho(x, b)]. \quad (2.7)$$

where each $T_{\mu\nu}(x, b)$ satisfies equations (1.3).

The vanishing of the trace $T_\mu^\mu(x, b)$ follows immediately from eq. (2.6). It is also straightforward to check that the equation $\partial_\nu T^{\mu\nu}(x, b) = 0$ is satisfied, where the derivative is taken with respect to x . Thus, conformal transformations generate from any solution of (1.3) a four-parameter family of solutions. Consider now the relativistic conformal hydrodynamics, where the stress-energy tensor is defined by the four fields: u^μ and T . The analysis above implies that special conformal transformations of any solution u^μ, T of the

conformal hydrodynamic equations generate a four-parameter family of solutions $T(x, b)$, $u_\mu(x, b)$, where

$$u_\mu(x, b) = (1 + 2b \cdot x + b^2 x^2) (\partial_\mu \Phi^\alpha) u_\alpha [\Phi(x, b)], \quad (2.8)$$

$$T(x, b) = \frac{T[\Phi(x, b)]}{1 + 2b \cdot x + b^2 x^2}. \quad (2.9)$$

One can verify that $u_\mu(x, b)$ has the correct normalization $u_\mu(x, b)u_\nu(x, b)\eta^{\mu\nu} = -1$ and the above transformations of u and T lead to transformation (2.7) both for the ideal hydrodynamics T^μ given by eq. (1.5) and for viscous hydrodynamics $T^{\mu\nu}$ given by eq. (1.6). Note that the velocity transformation (2.8) can be rewritten as

$$u^\mu(x, b) = u^\mu[\Phi(x, b)] + 2u^\alpha[\Phi(x, b)] \times \frac{b_\alpha x^\mu - b^\mu x_\alpha + 2b_\alpha x^\mu b \cdot x - b^2 x_\alpha x^\mu - x^2 b_\alpha b^\mu}{1 + 2b \cdot x + b^2 x^2}. \quad (2.10)$$

2.2 Special conformal symmetries of non-relativistic motions

In order to analyze the non-relativistic limit of special conformal transformations of the hydrodynamics it is convenient to consider the special conformal transformation of \mathbf{v} defined by $u_\alpha = (-\gamma, \gamma \mathbf{v}/c)$. Using $v_i = -cu_i/u_0$, one finds that eq. (2.8) implies the transformation of \mathbf{v}

$$v_i(x, b) = \frac{(1 + 2b \cdot x + b^2 x^2)v_i[\Phi(x, b)] + 2v_j[\Phi(x, b)] [b^j x_i - b_i x^j + 2b^j x_i b \cdot x - b^2 x^j x_i - x^2 b^j b_i]}{1 + 2b \cdot x + b^2 x^2 + 2[2b^0 x_0 b \cdot x + b^2 x_0^2 + x^2 b_0^2] - 2v_j[\Phi(x, b)] [b^j x_0 - b_0 x^j + 2b^j x_0 b \cdot x - b^2 x^j x_0 - x^2 b^j b_0]/c} \\ - \frac{2c[b^0 x_i - b_i x^0 + 2b^0 x_i b \cdot x - b^2 x^0 x_i - x^2 b^0 b_i]}{1 + 2b \cdot x + b^2 x^2 + 2[2b^0 x_0 b \cdot x + b^2 x_0^2 + x^2 b_0^2] - 2v_j[\Phi(x, b)] [b^j x_0 - b_0 x^j + 2b^j x_0 b \cdot x - b^2 x^j x_0 - x^2 b^j b_0]/c}. \quad (2.11)$$

Consider solutions of relativistic conformal hydrodynamics, where \mathbf{v} remains finite in the limit $c \rightarrow \infty$, and thus obeys the non-relativistic incompressible Euler (Navier-Stokes) equations. Then, if we employ in the above transformation such parameters b that $\mathbf{v}(x, b)$ is also finite at $c \rightarrow \infty$, we find a symmetry of the Euler equation. First it is necessary that the argument $\Phi(x, b)$ is finite in the limit $c \rightarrow \infty$. This implies $b^i = a^i/c^2$ and $b^0 = a/c$ where a^i and a have either finite or zero limit at $c \rightarrow \infty$. Thus,

$$t \rightarrow \frac{t}{1 - at}, \quad x^i \rightarrow \frac{x^i - a^i t^2}{(1 - at)^2}. \quad (2.12)$$

The expression (2.11) gives the transformation of \mathbf{v} in the non-relativistic limit

$$\mathbf{v}(\mathbf{x}, t, a, \mathbf{a}) = \mathbf{v} \left[\frac{t}{1 - at}, \frac{\mathbf{x} - \mathbf{a}t^2}{(1 - at)^2} \right] + 2 \frac{\mathbf{a}t - \mathbf{a}\mathbf{x}}{1 - at}. \quad (2.13)$$

If we take $a = 0$ (or equivalently assume that a scales as some negative power of c), while a_i is finite at $c \rightarrow \infty$, then we get the following symmetry of the Euler equation

$$\mathbf{v}(\mathbf{x}, t, \mathbf{a}) = \mathbf{v}[\mathbf{x} - \mathbf{a}t^2, t] + 2\mathbf{a}t. \quad (2.14)$$

This symmetry, discussed also in [11], corresponds to adding a uniform gravitational field to the Euler equation. Indeed, if $\mathbf{v}(\mathbf{x}, t)$ satisfies the incompressible Euler equation then $\mathbf{v}(\mathbf{x}, t, \mathbf{a})$ satisfies $\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + 2\mathbf{a}$. However $2\mathbf{a} = \nabla(2\mathbf{a} \cdot \mathbf{r})$ can be reabsorbed in the pressure, so that $\mathbf{v}(\mathbf{x}, t, \mathbf{a})$ is also a solution of the incompressible Euler equation. Thus, this symmetry corresponds to a combination of a passage to a uniformly accelerating frame and the possibility of reabsorbing a (uniform) gravitational field into the pressure (known in fluid mechanics see e.g. pp. 16 – 17 in [13]). The above consideration also works for the Navier-Stokes equation.

If we choose $a^i = 0$ (equivalent to assuming that a_i decreases as a negative power of c), while a remaining finite at $c \rightarrow \infty$ we get

$$\mathbf{v}(\mathbf{x}, t, a) = \mathbf{v} \left[\frac{t}{1-at}, \frac{\mathbf{x}}{(1-at)^2} \right] - \frac{2a\mathbf{x}}{1-at}. \tag{2.15}$$

This transformation describes an addition of a total expansion (from the center at $\mathbf{x} = 0$) or a total contraction (to the center at $\mathbf{x} = 0$) to the given flow. However, this transformation does not preserve the condition of finite velocity at large distances and as a result is not a symmetry of the incompressible Euler equation. In particular, out of an incompressible velocity it produces a field with $\nabla \cdot \mathbf{v}(\mathbf{x}, t, a) = 6a/(1-at)$.

Although the transformation (2.15) breaks the finiteness of velocity at infinity, and does not describe a symmetry of the incompressible Euler equation, it is a symmetry of the more general equation (1.11). Note, also that this transformation is different from the special conformal transformation that completes the Schrödinger symmetry group. The latter acts as

$$t \rightarrow \frac{t}{1-at}, \quad x^i \rightarrow \frac{x^i}{1-at}. \tag{2.16}$$

3 Exact solutions of ideal CFT hydrodynamics

3.1 One-dimensional motions and finite time singularities in ideal CFT hydrodynamics

Equations (1.3) can be written as [9]

$$\mathcal{D}\xi = -\frac{1}{3}\partial_\nu u^\nu, \quad \mathcal{D}u^\mu = -\partial^\mu \xi + \frac{u^\mu \partial_\nu u^\nu}{3}, \tag{3.1}$$

where $\xi \equiv \ln T$ and $\mathcal{D} = u^\alpha \partial_\alpha$. The first equation follows from the second by multiplication with u_μ and the use of $u^\mu u_\mu = -1$.

Let us consider one-dimensional hydrodynamic motions, that is motions that depend only on one spatial coordinate, say x , and have $u^y = u^z = 0$. It should be stressed that the microscopic dynamics of the system is still $(3+1)$ -dimensional, and it is only the macroscopic motion that is one-dimensional. For example, one could think of pushing the CFT fluid through a pipe where x is the coordinate along the pipe. It is convenient to parameterize the velocity by $u^0 = \cosh \phi$ and $u^x = \sinh \phi$. Substituting $u^\mu = (\cosh \phi, \sinh \phi, 0, 0)$,

we find that the $\mu = 0$ and $\mu = 1$ components of the second of eqs. (3.1) give

$$\frac{\partial \xi}{\partial t} = \frac{\sinh 2\phi}{3} \frac{\partial \phi}{\partial t} + c \frac{\cosh 2\phi - 2}{3} \frac{\partial \phi}{\partial x}, \quad (3.2)$$

$$-\frac{\partial \xi}{\partial x} = \frac{\cosh 2\phi + 2}{3c} \frac{\partial \phi}{\partial t} + \frac{\sinh 2\phi}{3} \frac{\partial \phi}{\partial x}, \quad (3.3)$$

respectively. This system has a more symmetric form when written as an evolution equation for ξ and ϕ ,

$$\frac{\partial \xi}{\partial t} = -\frac{c \sinh(2\phi)}{[\cosh(2\phi) + 2]} \frac{\partial \xi}{\partial x} - \frac{c}{[\cosh(2\phi) + 2]} \frac{\partial \phi}{\partial x}, \quad (3.4)$$

$$\frac{\partial \phi}{\partial t} = -\frac{c \sinh(2\phi)}{[\cosh(2\phi) + 2]} \frac{\partial \phi}{\partial x} - \frac{3c}{[\cosh(2\phi) + 2]} \frac{\partial \xi}{\partial x}. \quad (3.5)$$

It is interesting to notice the particular form that the above equations take, when considered up to quadratic order in ϕ . Introducing $v'/c = 2\phi/3$ and $\rho = T^2$ we find

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v')}{\partial x} = 0, \quad (3.6)$$

$$\rho \left(\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial x} \right) = -\frac{\partial p}{\partial x}, \quad p \equiv \frac{\rho c^2}{3}. \quad (3.7)$$

These equations describe an isothermal one-dimensional flow of gas with temperature $c^2/3$. They also describe one-dimensional isentropic, i.e constant entropy density, motion of a barotropic gas with velocity v' and pressure $p = \rho c^2/3$. In general, a barotropic gas with isentropic motion obeys $p = A\rho^\gamma$ where A is a constant and $\gamma \geq 1$ is the polytropic exponent of the gas, which is equal to the ratio of the heat capacities at constant pressure and volume, respectively. Here $\gamma = 1$, which corresponds to gas in the limit of a large number of internal degrees of freedom of the molecules. It is notable that a motion of such a gas admits a gravity dual interpretation, as we will discuss in the last section.

The system of equations (3.2)–(3.3) is a 2×2 system of quasi-linear PDE. It can be recast using Riemann variables, which are conserved along characteristic directions in space [14]. Here the result of the procedure can be readily seen directly from the equations. Multiplying the second of eqs. (3.4)–(3.5) with $1/\sqrt{3}$ and adding and subtracting it from the first equation we get

$$\frac{\partial \xi}{\partial t} + \frac{c [\sinh 2\phi \pm \sqrt{3}]}{\cosh 2\phi + 2} \frac{\partial \xi}{\partial x} \pm \frac{1}{\sqrt{3}} \left[\frac{\partial \phi}{\partial t} + \frac{c [\sinh 2\phi \pm \sqrt{3}]}{\cosh 2\phi + 2} \frac{\partial \phi}{\partial x} \right] = 0. \quad (3.8)$$

These equations relate derivatives of ϕ and ξ along the two families $x_\pm(t)$ of characteristic directions, defined by

$$\frac{dx_\pm}{dt} = \frac{c [\sinh 2\phi \pm \sqrt{3}]}{\cosh 2\phi + 2}. \quad (3.9)$$

Thus, two families of characteristics emanate from each point of the fluid. They describe non-linear propagation of sound to the right (x_+) and to the left (x_-). In the limit of linear

sound (small ϕ), the equations are simply $\dot{x}_{\pm} = \pm c/\sqrt{3}$ where $c/\sqrt{3}$ is the speed of sound. Correspondingly, the Riemann variables have a remarkably simple form

$$r_{\pm} = \xi \pm \frac{\phi}{\sqrt{3}}, \tag{3.10}$$

and the equations governing one-dimensional hydrodynamic motions of the CFT fluid can be written in the form

$$\left[\frac{\partial}{\partial t} + c \frac{\sinh [\sqrt{3}(r_+ - r_-)] + \sqrt{3}}{\cosh [\sqrt{3}(r_+ - r_-)] + 2} \frac{\partial}{\partial x} \right] r_+ = 0, \tag{3.11}$$

$$\left[\frac{\partial}{\partial t} + c \frac{\sinh [\sqrt{3}(r_+ - r_-)] - \sqrt{3}}{\cosh [\sqrt{3}(r_+ - r_-)] + 2} \frac{\partial}{\partial x} \right] r_- = 0. \tag{3.12}$$

The above system admits special solutions, called simple waves, where one of Riemann variables is a constant. For example, for $r_- = \text{const}$, we find a closed equation on r_+ of the form

$$\partial_t r_+ + f(r_+) \partial_x r_+ = 0. \tag{3.13}$$

Introducing $w = f(r_+)$ we find that w obeys the Hopf equation

$$\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial x} = 0. \tag{3.14}$$

The equation describes formation of discontinuities (shocks) in w in a finite time. The shocks form when the spatial derivative of the initial condition on w is negative somewhere. Indeed, introducing $\sigma = \partial_x w$ we find that σ obeys

$$\frac{d}{dt} \sigma[x(t), t] = -\sigma^2[x(t), t], \quad \frac{dx}{dt} = w[x(t), t]. \tag{3.15}$$

This equation predicts finite time blowup of σ , that occurs at $x(t)$ emanating from the initial point where σ has the (negative) minimum. It is easy to see that the blowup corresponds to the formation of a discontinuity in w . One can use the implicit form of the solution of eq. (3.14),

$$w(x, t) = w_0 [x - w(x, t)t], \quad w_0 \equiv w(x, t = 0). \tag{3.16}$$

Qualitatively, eq. (3.14) describes velocity field of free particles moving on a line where the discontinuity forms when faster particles catch up with the slower particles in front of them.

The formal reason for the formation of discontinuities in a simple wave is the intersection of characteristics. The field w in eq. (3.14) is conserved along the characteristic lines $x(t)$ defined in eq. (3.15). In general, nothing prevents the lines from intersecting in a finite time T . At $t > T$ the solution obtained by imposing the field conservation along the characteristics would become multi-valued, which is signaled by the divergence of the spatial derivative of w at the intersection point. The same property of the finite-time formation of infinite derivatives of the fields holds for the solutions of the system (3.11)–(3.12) and thus,

also of the original system (3.2)–(3.3). Indeed, the equations imply that r_{\pm} are conserved along the corresponding characteristics in (x, t) plane. Generically, these characteristics intersect in a finite time, and at the intersection time a field derivative diverges. Thus, we conclude that ideal hydrodynamics equations in one dimension lead to a finite time singularity of the derivatives.

3.2 Special conformal symmetry generation of higher-dimensional solutions

As shown in the previous section, one may solve rather generally the one-dimensional dynamics of the fluid in a CFT hydrodynamics. We can generate from the one-dimensional solutions, higher dimensional ones by applying special conformal transformation as in eqs. (2.8)–(2.9). With the help of eq. (2.10), one finds that a solution of one-dimensional hydrodynamics leads to a family of solutions:

$$\begin{aligned}
 u^0(x, b) &= \cosh \phi [\Phi(x, b)] \left[1 + 2 \frac{(\mathbf{b}x^0 - \mathbf{x}b^0)^2}{1 + 2\mathbf{b} \cdot \mathbf{x} + b^2x^2} \right] + \frac{2 \sinh \phi [\Phi(x, b)]}{1 + 2\mathbf{b} \cdot \mathbf{x} + b^2x^2} \\
 &\quad \times \left[b_1x^0 - b^0x_1 + 2b_1x^0b \cdot \mathbf{x} - b^2x_1x^0 - x^2b_1b^0 \right], \\
 u^1(x, b) &= \sinh \phi [\Phi(x, b)] \left[1 + 2 \frac{2b^1x^1b \cdot \mathbf{x} - b^2(x^1)^2 - x^2(b^1)^2}{1 + 2\mathbf{b} \cdot \mathbf{x} + b^2x^2} \right] + 2 \cosh \phi [\Phi(x, b)] \\
 &\quad \times \frac{b_0x^1 - b^1x_0 + 2b_0x^1b \cdot \mathbf{x} - b^2x_0x^1 - x^2b_0b^1}{1 + 2\mathbf{b} \cdot \mathbf{x} + b^2x^2}, \tag{3.17}
 \end{aligned}$$

$$\begin{aligned}
 u^2(x, b) &= \frac{2}{1 + 2\mathbf{b} \cdot \mathbf{x} + b^2x^2} \left[\cosh \phi [\Phi(x, b)] \left(b_0x_2 - b_2x_0 + 2b_0x_2b \cdot \mathbf{x} - b^2x_0x_2 - x^2b_0b_2 \right) \right. \\
 &\quad \left. + \sinh \phi [\Phi(x, b)] \times (b_1x_2 - b_2x_1 + 2b_1x_2b \cdot \mathbf{x} - b^2x_1x_2 - x^2b_1b_2) \right] \\
 \xi(x, b) &= \xi [\Phi(x, b)] + \ln [1 + 2\mathbf{b} \cdot \mathbf{x} + b^2x^2], \tag{3.18}
 \end{aligned}$$

where $b^\mu = (b^0, \mathbf{b})$ and $x^\mu = (x^0, \mathbf{x})$, the expression for u^3 is obtained from the expression for u^2 by interchange of indices, and ϕ and ξ are obtained from the one-dimensional hydrodynamics.

3.3 The non-relativistic limit

The relativistic one-dimensional solution does not have a non-trivial slow motions limit since the incompressibility condition implies that the velocity is independent of the coordinate x . This is in accord with the presence of a factor of c in front of the space derivative in eqs. (3.11)–(3.12). We have seen that we can generate from the relativistic one-dimensional solution, higher dimensional ones by applying special conformal transformations. It is still natural to ask whether these higher-dimensional solutions have a non-trivial slow motions limit, i.e. whether they also solutions of the incompressible Navier-Stokes equations. We will show in the following that this is not the case, and also the higher-dimensional solutions do not have a non-trivial slow motions limit. Of course, this is not really a surprise,

since as we noted above, the one-dimensional solution with which we started in order to generate the higher-dimensional ones, has a trivial limit.

For simplicity we set the time coordinate to zero. One can write the general structure of the velocity v^i using equations (3.18) as

$$v_i(r) = (1 + 2b \cdot \mathbf{x} + b^2 \mathbf{x}^2) \phi(\Phi) \nabla_i \Phi, \tag{3.19}$$

where

$$\Phi = \frac{x + b^x \mathbf{x}^2}{1 + 2b \cdot \mathbf{x} + b^2 \mathbf{x}^2}. \tag{3.20}$$

The incompressibility condition implies then that

$$\frac{\phi'}{\phi}(\Phi) = -\frac{\nabla_i [(1 + 2b \cdot \mathbf{x} + b^2 \mathbf{x}^2) \nabla_i \Phi]}{(1 + 2b \cdot \mathbf{x} + b^2 \mathbf{x}^2)(\nabla \Phi)^2}. \tag{3.21}$$

Since the l.h.s. of (3.21) is a function of Φ , we need to require that the r.h.s. is also a function of Φ in order to be able to solve for ϕ . However, a straightforward calculation shows that the r.h.s. of (3.21) is not a function of Φ , for Φ finite in the slow motions limit. Thus, the incompressibility equation cannot be solved, and there is no non-trivial slow motion limit to the higher-dimensional solution of the relativistic conformal hydrodynamics.

4 Shock solutions

4.1 Shocks-global solutions in ideal hydrodynamics

We have seen that the relativistic ideal conformal hydrodynamics evolution in one dimension produces discontinuities in a finite time. The mechanism for the generation of the singularity is the intersection of characteristics, and it holds in higher-dimensional motions as well. A simple concrete example is the multi-dimensional generalization of the Hopf equation (3.14) provided by $\partial_t \mathbf{w} + \mathbf{w} \nabla \mathbf{w} = 0$. The matrix $\sigma_{ij} = \partial_j w_i$ obeys the equation $d\sigma/dt = -\sigma^2$, that predicts a finite time blow up of σ . The finite-time generation of discontinuities is a general property of ideal hydrodynamics.

A natural question that arises is how should we define the hydrodynamics evolution beyond the singularity time? One way is to include the viscous contribution to the stress tensor, and consider eqs. (1.3) and (1.6) instead of eqs. (1.3) and (1.5). According to the ideal hydrodynamics equations, the viscous contribution grows in time and becomes infinite at the time of formation of the discontinuity [cf. eq. (1.7)]. Thus, the viscous contribution is important near the singularity. It is conjectured (cf. [5, 6]), that this contribution regularizes the singularity, makes all the fields finite and provides a well-defined evolution within the frame of differential equations.

A different approach for dealing with the singularity is to continue the solutions of the ideal hydrodynamics beyond the singularity time, by passing from classical (differentiable) to weak solutions of the equations, where weak solutions are solutions with discontinuities. This formulation is very useful when considering high Reynolds number flows, for which the viscosity is effectively small [6]. In such flows the viscosity is important only near the

points, where a discontinuity would form in the absence of viscosity. The effect of viscosity is to smooth the discontinuity, while in the other regimes of the flow the viscosity effect is small.

An appropriate framework is to apply an ideal hydrodynamics evolution everywhere, except at the hypersurfaces where a discontinuity forms. At the discontinuity one requires that the normal components of fluxes of locally conserved energy and momentum densities remain continuous. This requirement is necessary in order for the equations to describe the local conservation laws, as the hydrodynamics should do. The corresponding constraints at the discontinuity are called the Rankine-Hugoniot conditions.

The main advantage of the obtained weak solutions - the shocks - is that they provide within the framework of ideal hydrodynamics, global and valid solutions at all times. Let us construct a simple example of such a solution. The simplest solution of the system of eqs. (1.3) and (1.5) is a boosted equilibrium state, where T and u^μ are constants. We can saw two equilibrium solutions along a plane, and form a simple but non-trivial solution - a shock wave. T and u^μ are piecewise constant and they have a single discontinuity along a plane $x = x(t)$. In the frame that moves with the shock where $x(t) \equiv 0$ (the considered shock solutions move at the constant speed), the fluxes of the conserved quantities must be continuous through the shock. This leads to the Rankine-Hugoniot conditions

$$[T^4 (\gamma^{\mu x} + 4u^\mu u^x)] = 0, \tag{4.1}$$

where the brackets stand for the value of the jump in the function through the shock. In contrast, the entropy flux does not have to be continuous - rather the entropy grows in the shock.

Using $u^\mu = (\gamma, \gamma \mathbf{v})$ (here we set $c = 1$), we get from eq. (4.1)

$$\left[\frac{T^4 v_x}{1 - v^2} \right] = 0, \quad \left[T^4 + \frac{4T^4 v_x^2}{1 - v^2} \right] = 0, \tag{4.2}$$

and

$$\left[\frac{T^4 v_y v_x}{1 - v^2} \right] = 0, \quad \left[\frac{T^4 v_z v_x}{1 - v^2} \right] = 0. \tag{4.3}$$

If v_x is non-zero the equations imply that the velocity components parallel to the shock plane are continuous across the shock. For convenience, we can set these components to zero, and we are left with

$$\left[\frac{T^4 v_x}{1 - v_x^2} \right] = 0, \quad \left[T^4 + \frac{4T^4 v_x^2}{1 - v_x^2} \right] = 0. \tag{4.4}$$

Note, that if we use the general conditions for the shock [6] with $p = aT^4$ and heat function $w = 4aT^4$ (where $a = const$), then we get the same conditions.

If we denote by T_1, T_2 and v_1, v_2 the values of the temperature and the velocity component v_x at two sides of the shock, then the conditions read

$$\frac{T_2^4 v_2}{1 - v_2^2} = \frac{T_1^4 v_1}{1 - v_1^2}, \quad T_2^4 + \frac{4T_2^4 v_2^2}{1 - v_2^2} = T_1^4 + \frac{4T_1^4 v_1^2}{1 - v_1^2}. \tag{4.5}$$

These equations define a two-parameter family of shock solutions of hydrodynamics. The two parameters correspond to the values of the fluxes of the energy and the x -component of the momentum through the shock. Evidently, $T_1 = T_2$ and $v_1 = v_2$ solve the above equations. In order to analyze the conditions for a non-trivial solution, we assume that the fields T_2, v_2 at $x < 0$ are given (one can assume $v_2 > 0$ as the solution with $v_2 < 0$ can be obtained by the sign reversal). Equating the expressions for T_2^4/T_1^4 , we find a quadratic equation for v_1 that has two solutions, $v_1 = \frac{1}{3v_2}$ and the trivial solution $v_1 = v_2$. Thus, a non-trivial solution with $v_1 = \frac{1}{3v_2} \leq 1$ exists only for v_2 above the threshold value of $1/3$ (recall that the speed of sound is $1/\sqrt{3}$). In the frame for which the fluid is at rest at $-\infty$, the shock always moves at a speed larger than $1/3$. Note that the difference $v_1 - v_2$ can be arbitrarily small.

Fixing the temperature from eq. (4.5), we find that non-trivial shock solutions at $v_2 > 1/3$ are described by the relations

$$v_1 = \frac{1}{3v_2}, \quad T_1^4 = T_2^4 \frac{9v_2^2 - 1}{3(1 - v_2^2)}. \quad (4.6)$$

It is remarkable that the Rankine-Hugoniot conditions for the CFT have such explicit solutions, cf. the usual situation [6]. Note that $v_1 > v_2$ for $1/3 < v_2 < 1/\sqrt{3}$ while $v_1 < v_2$ for $v_2 > 1/\sqrt{3}$ where $1/\sqrt{3}$ is the speed of sound. For v_2 close to the speed of sound we have $v_1 \approx v_2$ while at the exact equality $v_2 = 1/\sqrt{3}$ the solution trivializes. Finally v_1 , like v_2 , belongs to the range $1/3 < v_1 < 1$ as should be due to the symmetry in the indices interchange.

The above shock solutions are stationary non-equilibrium states of the fluid with finite dissipation in the shock. The stationarity is sustained by the fact that the solution is non-vanishing at infinity, and the presence of finite fluxes in the system.

4.2 Special conformal transformation of shocks

Shocks are quite simple solutions of ideal hydrodynamics: they are time-independent, one-dimensional and piecewise constant. As a result they allow a clear insight into what kind of a change the special conformal transformations produce in the solution. Using eqs. (2.11) and (2.9) we find that shocks produce the following weak solution of the hydrodynamics

$$v_i(x, b) = \frac{(1 + 2b \cdot x + b^2 x^2)v_{[k]}\delta_{i1} + 2v_{[k]}[b^1 x_i - b_i x^1 + 2b^1 x_i b \cdot x - b^2 x^1 x_i - x^2 b^1 b_i]}{1 + 2b \cdot x + b^2 x^2 + 2[2b^0 x_0 b \cdot x + b^2 x_0^2 + x^2 b_0^2] - 2v_{[k]}[b^1 x_0 - b_0 x^1 + 2b^1 x_0 b \cdot x - b^2 x^1 x_0 - x^2 b^1 b_0] / c} - \frac{2c[b^0 x_i - b_i x^0 + 2b^0 x_i b \cdot x - b^2 x^0 x_i - x^2 b^0 b_i]}{1 + 2b \cdot x + b^2 x^2 + 2[2b^0 x_0 b \cdot x + b^2 x_0^2 + x^2 b_0^2] - 2v_{[k]}[b^1 x_0 - b_0 x^1 + 2b^1 x_0 b \cdot x - b^2 x^1 x_0 - x^2 b^1 b_0] / c}, \quad (4.7)$$

$$T(x, b) = \frac{T_{[k]}}{1 + 2b \cdot x + b^2 x^2}. \quad (4.8)$$

where $[k] = 1, 2$ and $v_{[k]}, T_{[k]}$ are some constants obeying eqs. (4.5). The equation of discontinuity hyperplane (where one should switch $k = 1$ to $k = 2$ in the above formula) is given by $x^1 + b^1 x \cdot x = 0$. One observes that the special conformal transformations allow to generate quite complicated motions out of even relatively simple motions.

4.3 Shocks in viscous CFT hydrodynamics

The ideal hydrodynamics shock solution constructed above is clearly not uniformly valid in space. It contains a discontinuity region, where derivatives are formally infinite indicating that one must account for the viscous, and maybe also higher order derivatives contributions to $T^{\mu\nu}$. The ideal hydrodynamics shock describes correctly the exact solution far from the region of fast variations of the fields, and it arises when the formal limit of zero viscosity is taken in the exact solution. Here we provide the exact solution of the viscous hydrodynamics, and demonstrate how the one-dimensional shock considered in the first subsection arises in the limit.

The stationary one-dimensional solutions of the hydrodynamics equations satisfy

$$\partial_\nu T^{\mu\nu} = \partial_0 T^{\mu 0} + \partial_x T^{\mu x} = 0, \quad \partial_0 T^{\mu 0} = 0. \quad (4.9)$$

These equations imply that

$$T^{0x} = C_1, \quad T^{xx} = C_2, \quad (4.10)$$

where C_1 and C_2 are the constant fluxes of energy and momentum, respectively.

As already mentioned above, the states in question are non-equilibrium steady states sustained by the presence of finite fluxes in the system. In an ideal hydrodynamics, the non-trivial solutions of the above equations with different values of velocity and temperature at $x = -\infty$ and $x = +\infty$, are obtained by introducing a discontinuity at $x = 0$. In contrast, in the viscous hydrodynamics the different asymptotic values at $x = -\infty$ and $x = +\infty$ are connected smoothly and the discontinuity at $x = 0$ changes to a smooth transition from the values of the fields at $x = -\infty$ to those at $x = +\infty$. Note, that at large $|x|$ the derivatives are small and ideal hydrodynamics is a good approximation. The regions that are influenced by the viscosity are at the core of the shock.

In order to take into account viscosity, we consider $\sigma_{\mu\nu}$ in eqs. (1.6)–(1.7). One notes that the time-derivatives appearing in $\sigma_{\mu\nu}$ can be substituted by their ideal hydrodynamics expressions. Indeed, within hydrodynamics the time derivatives of fields are given by a series in the spatial gradients of the fields. The second derivative terms in $\sigma_{\mu\nu}$ are higher order terms in the expansion compared to the considered order involving the first derivatives [15]. The use of eqs. (3.1) allows to rewrite eq. (1.7) as [10]

$$\sigma_{\mu\nu} = \partial_\mu u_\nu + \partial_\nu u_\mu - u_\nu \partial_\mu \xi - u_\mu \partial_\nu \xi - 2\eta_{\mu\nu} \partial_\alpha u^\alpha / 3,$$

which gives

$$\sigma_{0x} = \cosh \phi \partial_t \phi - \sinh \phi \partial_x \phi + \cosh \phi \partial_x \xi - \sinh \phi \partial_t \xi, \quad (4.11)$$

$$\sigma_{xx} = 4 \cosh \phi \partial_x \phi / 3 - 2 \sinh \phi \partial_x \xi - 2 \sinh \phi \partial_t \phi / 3. \quad (4.12)$$

The above expressions contain time-derivatives that make the equation $\partial_\nu T^{\mu\nu} = 0$ a second order equation in the time derivative. In order to obtain a well-defined evolution

equation, which is first-order in time, one has to use the ideal hydrodynamics equation and express the time derivatives via space derivatives. Using eqs. (3.4)–(3.5) one finds

$$\sigma_{0x} = -\frac{2w \sinh 2\phi}{2 + \cosh 2\phi}, \quad \sigma_{xx} = \frac{2w[1 + \cosh 2\phi]}{2 + \cosh 2\phi}, \quad (4.13)$$

$$w = \cosh \phi \frac{\partial \phi}{\partial x} - \sinh \phi \frac{\partial \xi}{\partial x} = T \frac{\partial}{\partial x} \left(\frac{\sinh \phi}{T} \right). \quad (4.14)$$

Note, that both components involve the same combination of derivatives w . This combination contains besides the derivative of the velocity also the derivative of the temperature. In non-relativistic hydrodynamics such terms arise from the heat conduction. Here their appearance is in accord with the fact that the fluid viscosity allows to capture in a unified way both the effects of viscosity and the heat conduction. With viscosity the components of the stress-energy tensor read

$$T^{0x} = 2T^4 \sinh 2\phi - \frac{2F(\lambda)T^4 \sinh 2\phi}{2 + \cosh 2\phi} \frac{\partial}{\partial x} \left(\frac{\sinh \phi}{T} \right), \quad (4.15)$$

$$T^{xx} = T^4 [2 \cosh 2\phi - 1] - \frac{2F(\lambda)T^4 [1 + \cosh 2\phi]}{2 + \cosh 2\phi} \frac{\partial}{\partial x} \left(\frac{\sinh \phi}{T} \right). \quad (4.16)$$

Thus, stationary one-dimensional solutions of the CFT hydrodynamics obey (cf. eqs. (4.10) and note that partial derivatives becomes ordinary ones):

$$2T^4 \sinh 2\phi - \frac{2F(\lambda)T^4 \sinh 2\phi}{2 + \cosh 2\phi} \frac{d}{dx} \left(\frac{\sinh \phi}{T} \right) = C_1, \quad (4.17)$$

$$T^4 [2 \cosh 2\phi - 1] - \frac{2F(\lambda)T^4 [1 + \cosh 2\phi]}{2 + \cosh 2\phi} \frac{d}{dx} \left(\frac{\sinh \phi}{T} \right) = C_2. \quad (4.18)$$

These equations pass to the ideal hydrodynamics equations in the zero viscosity limit $F(\lambda) \rightarrow 0$. Since far from $x = 0$ the viscous contribution in these equations can be neglected due to the smallness of the derivatives, we find that the asymptotic values of the functions at $+\infty$ and $-\infty$ satisfy

$$\begin{aligned} T_{+\infty}^4 \sinh 2\phi_{+\infty} &= T_{-\infty}^4 \sinh 2\phi_{-\infty} = C_1/2 \\ T_{+\infty}^4 [2 \cosh 2\phi_{+\infty} - 1] &= T_{-\infty}^4 [2 \cosh 2\phi_{-\infty} - 1] = C_2. \end{aligned} \quad (4.19)$$

These conditions with $v = \tanh \phi$ correspond to the Rankine-Hugoniot conditions (4.5). In this way the shock characteristics T_i and v_i in eq. (4.5) describe the asymptotic values of the fields far from the region, where viscosity is important. In order to construct the values of the fields in the viscous region, which corresponds to a discontinuity in the shock description, one has to solve the system of eqs. (4.17)–(4.18). Multiplying the first equation by $1 + \cosh 2\phi$, the second by $\sinh 2\phi$, and subtracting the equations we find

$$T^4 = \frac{C_1 \coth \phi - C_2}{3}. \quad (4.20)$$

Plugging the above relation into eq. (4.17) we find that ϕ satisfies

$$\frac{2 \sinh 2\phi [C_1 \coth \phi - C_2]}{3} \left[1 - \frac{3^{1/4} F(\lambda)}{2 + \cosh 2\phi} \frac{d}{dx} \left(\frac{\sinh \phi}{[C_1 \coth \phi - C_2]^{1/4}} \right) \right] = C_1. \quad (4.21)$$

We seek a solution to this equation that satisfies Neumann boundary conditions, i.e. vanishing of the derivatives, at $x = -\infty$ and $x = +\infty$. One can rewrite the equation as

$$\frac{d\phi}{dx} = \frac{T(\phi)[2 + \cosh 2\phi][2T^4(\phi) \sinh 2\phi - C_1]}{F(\lambda) \cosh \phi [2T^4(\phi) \sinh 2\phi + C_1/3]}, \quad (4.22)$$

with $T(\phi)$ defined by eq. (4.20). We find that $v = \tanh \phi$ obeys

$$\frac{dv}{dx} = \frac{3^{3/4} \zeta C_2^{1/4} [\zeta - v]^{1/4} [3 - v^2] [v - v_1] [v - v_2] \sqrt{1 - v^2}}{F(\lambda) v^{1/4} [\zeta(5 - v^2) - 4v]}, \quad (4.23)$$

where $\zeta \equiv C_1/C_2$ and $v_{1,2}$ are the roots of $\zeta(1 + 3v^2) - 4v = 0$,

$$v_{1,2} = \frac{2 \pm \sqrt{4 - 3\zeta^2}}{3\zeta}. \quad (4.24)$$

Note that $v_1 = 1/3v_2$, cf. eq. (4.6). The viscous counterpart of the ideal hydrodynamics shock considered in the first subsection corresponds to the range $1 \leq \zeta < 2/\sqrt{3}$ where $v_{1,2}$ are real and $1/3 \leq v_{1,2} \leq 1$.

We first consider the case $v(-\infty) < v(+\infty)$ and search for the solution of eq. (4.23) satisfying

$$v(-\infty) = v_2 = \frac{2 - \sqrt{4 - 3\zeta^2}}{3\zeta}, \quad v(+\infty) = v_1 = \frac{2 + \sqrt{4 - 3\zeta^2}}{3\zeta}. \quad (4.25)$$

Such a solution must have a positive derivative and values close to v_2 in the range $v_2 < v < v_1$ in some region of space. However, eq. (4.23) dictates that at least close to $v_{1,2}$ the sign of dv/dx is determined by $[v - v_1][v - v_2]$ which is negative at $v_2 < v < v_1$. This follows by observing that the factor $\zeta(5 - v^2) - 4v$ in the denominator of eq. (4.23) obeys $\zeta(5 - v^2) - 4v = 3\zeta[v - v_1][v - v_2] + 4\zeta[1 - v^2]$. Thus it is not possible to construct the solutions with $v(-\infty) < v(+\infty)$.

In contrast, the solutions with $v(-\infty) > v(+\infty)$ for which

$$v(-\infty) = v_2 = \frac{2 + \sqrt{4 - 3\zeta^2}}{3\zeta}, \quad v(+\infty) = v_1 = \frac{2 - \sqrt{4 - 3\zeta^2}}{3\zeta}. \quad (4.26)$$

are easily constructed. Noting that $\zeta(5 - v^2) - 4v$ is always positive for $v_2 < v < v_1$ at $\zeta > 1$ we conclude from eq. (4.23) that $v(x)$ decreases monotonically from $[2 + \sqrt{4 - 3\zeta^2}]/3\zeta$ at $x = -\infty$ to $[2 - \sqrt{4 - 3\zeta^2}]/3\zeta$ at $x = +\infty$. Here we use that it takes the solution of an equation of the type $dv/dt = H(v)$ infinite "time" t to reach a zero of an analytic function $H(\phi)$. The implicit form of the solution is given by

$$x - x_0 = \frac{F(\lambda)}{3^{3/4} \zeta C_2^{1/4}} \times \int_{2/3\zeta}^v \frac{v'^{1/4} [\zeta(5 - v'^2) - 4v'] dv'}{[\zeta - v']^{1/4} [3 - v'^2] [v' - [2 + \sqrt{4 - 3\zeta^2}]/3\zeta] [v' - [2 - \sqrt{4 - 3\zeta^2}]/3\zeta] \sqrt{1 - v'^2}}, \quad (4.27)$$

where the constant x_0 is the point at which $v(x)$ attains its middle value $2/3\zeta$ (naturally the solution is translation-invariant). In the following we set $x_0 = 0$. Note that $v(x, C_1, C_2) = v(C_1/C_2, C_2^{1/4}x)$. The dependence of temperature on x is determined by $T^4(x) = C_2[\zeta - v(x)]/3v(x)$. One observes that the necessary condition that the resulting expression for T^4 is positive is satisfied by $\zeta > 1$ and $v < 1$. The ideal hydrodynamics shock is recovered in the formal limit of zero viscosity, $F(\lambda) \rightarrow 0$.

Thus the viscosity selects among the ideal hydrodynamics shocks only those for which the inflow velocity $v(-\infty)$ is larger than the outflow velocity $v(+\infty)$. This has a simple meaning: the dropping of velocity corresponds to a dissipative action of the viscosity (friction), while velocity increase would qualitatively mean an accelerating action of the friction. The selected flows start as supersonic and end as subsonic.

It remains to consider the condition under which the above solution has a small Knudsen number to justify neglecting the higher derivative terms in T^{0x} and T^{xx} . We notice that generally $F(\lambda)$ is of the same order as $G(\lambda)$ appearing in the expression (1.2) for l_{cor} , see [10, 12]. We define the local Knudsen number by

$$Kn = l_{cor} \left| \frac{v'}{v} \right| = \frac{3\zeta[3 - v^2][v - v_1][v - v_2]\sqrt{1 - v^2}}{v[\zeta(5 - v^2) - 4v]}, \quad (4.28)$$

where we used eqs. (1.2) and (4.23). The viscous shock solution above is valid if $Kn \ll 1$ uniformly. At fixed ζ , by choosing a sufficiently small C_2 one may achieve an arbitrarily large scale of variations of v in space, due to $v(x, C_1, C_2) = v(C_1/C_2, C_2^{1/4}x)$. This, however, does not lead to a solution with a small Knudsen number, since the correlation length also increases under such scaling of C_i . This feature is taken into account by eq. (4.28), that depends only on the ratio C_1/C_2 .

To check that $Kn \ll 1$, it is enough to substitute in the above equation the middle value $2/3\zeta$ of v which produces a term of order $4 - 3\zeta^2$ (ζ belongs to the range $[1, 2/\sqrt{3}]$). It follows that $Kn \ll 1$ is satisfied only for ζ close to $2/\sqrt{3}$ (a more precise condition is derived below). For such ζ , the inflow velocity $v(-\infty)$ is close to the speed of sound $1/\sqrt{3}$, and the outflow obeys $v(+\infty) \approx v(-\infty)$, cf. remarks to eq. (4.6). The Knudsen number is small just because the total variation of v is small. In other words, for this solution the requirement of locally small variations of fields, implies also that the global variation is small.

4.4 The viscous shock solution at a small Knudsen number

In the small Knudsen number domain $\zeta = 2/\sqrt{3} - \epsilon$ with $\epsilon \ll 1$, the velocity changes in the interval $[1/\sqrt{3} - \delta, 1/\sqrt{3} + \delta]$ where $\delta = \epsilon^{1/2}3^{-1/4}$. Using that the total variation of the solution is small, one can obtain more explicit expressions in this case. To leading order in δ eq. (4.23) becomes

$$\frac{d\tilde{v}}{dx} = \frac{3^{1/4}\sqrt{2}C_2^{1/4}[\tilde{v} - \delta][\tilde{v} + \delta]}{F(\lambda)}, \quad (4.29)$$

where $\tilde{v} = v - 1/\sqrt{3}$ is the velocity deviation from the speed of sound. The solution of the above equation gives the well-known kink solution, which can also be obtained by

expanding the integral (4.27),

$$v = \frac{1}{\sqrt{3}} - \delta \tanh \left[\frac{x}{L} \right], \quad L = \frac{F(\lambda)}{3^{1/4} \sqrt{2} C_2^{1/4} \delta} \sim \frac{l_{cor}}{\delta}, \quad (4.30)$$

where we used that $T \approx [C_2/3]^{1/4}$ holds for the solution. One has

$$T^4 \approx \frac{C_2}{3} + \frac{2C_2\delta}{\sqrt{3}} \tanh \left[\frac{x}{L} \right]. \quad (4.31)$$

We see that the Knudsen number $Kn = l_{cor}/L \sim \delta$ is small at $\delta \ll 1$. Note, that the Knudsen number definition (4.28) would give incorrectly $Kn \sim \delta^2$ - the reason is that $v(x)$ has a special form where $v'(x)$ is determined by the small correction \tilde{v} to the main constant part $1/\sqrt{3}$.

One can exclude the constant component in $v(x)$ by passing to the frame moving at the speed of sound, where the solution takes the form of a kink propagating at the sound speed,

$$\begin{aligned} \frac{v}{c} &= -\frac{3\delta}{2} \tanh \left[\sqrt{\frac{3}{2}} \left(\frac{x + ct/\sqrt{3}}{L} \right) \right], \\ T^4 &\approx \frac{C_2}{3} + \frac{2C_2\delta}{\sqrt{3}} \tanh \left[\sqrt{\frac{3}{2}} \left(\frac{x + ct/\sqrt{3}}{L} \right) \right], \end{aligned} \quad (4.32)$$

in the leading order in δ , where we restored c in v . One sees that the solution is a near equilibrium state. In the limit $c \rightarrow \infty$ the motion gets degenerated to $v = const$ in accord with the fact that the only incompressible one-dimensional flow is a constant one.

4.5 The shock solution at a Knudsen number of order unity

We saw above that if the shock solution is such that the inflow velocity $v(-\infty)$ differs significantly from the speed of sound then there are regions in the flow where $Kn \sim 1$ (this follows from self-consistency: if $Kn \ll 1$ then viscous hydrodynamics should work well). The solution of the viscous hydrodynamics then does not describe the solutions correctly in those regions though it may give a good idea of the solution there. To find an exact solution one should account for higher order derivatives in the constitutive relation for $T^{\mu\nu}$ (for instance, the second derivative terms were discussed in [9, 16]). The above however does not undermine the utility of the ideal hydrodynamics shock which is still a good description of the solution far from the transition region near $x = 0$.

We show this point explicitly. We consider a non-equilibrium steady state of a CFT fluid which is sustained by the fluxes of energy and x -component of momentum from infinity. In this state the mean value of $T^{\mu\nu}$ depends only on x and eqs. (4.10) are satisfied irrespective of the validity of the hydrodynamic constitutive relations. Now, far from the transition region the scale of variations of the mean value of the stress-energy tensor becomes large, the state is locally very close to equilibrium and the ideal hydrodynamics approximation $T_{\mu\nu} = T^4 [\eta_{\mu\nu} + 4u_\mu u_\nu]$ works with increasing degree of accuracy. In other words, far from the transition region the state looks locally closer and closer to the global equilibrium (the information that the local equilibrium state is not the global one becomes

further separated in space). As a result, the ideal hydrodynamics shock solution allows to establish correctly at all orders the connection between the asymptotic values of $v(-\infty)$, $T(-\infty)$ and $v(+\infty)$, $T(+\infty)$, and thus also of $T^{\mu\nu}(-\infty)$ and $T^{\mu\nu}(+\infty)$. As far as the intermediate region of the transition is concerned, it is described by the solution in the previous subsection for the inflow velocity close to the speed of sound $1/\sqrt{3}$. When the inflow velocity $1/\sqrt{3} < v(-\infty) < 1$ is not close to $1/\sqrt{3}$ the description of the transition region demands the account of the higher order derivatives. We do not expect, however, that the higher derivative terms will lead to a further selection of the ideal hydrodynamic shocks beyond the one imposed by the viscosity.

4.6 How is it possible to have $Kn \ll 1$, while the viscous term still being important?

It is natural to ask why is it possible to construct a solution where the viscous term, whose effect is generally measured by the Knudsen number, is important while higher order derivative terms, whose effect is measured by the same parameter, are not. The reason being the presence of an additional dimensionless parameter in the equations, v/c (here we restore c in the relations). As long as we consider relativistic motions with $v \sim c$, the viscous to ideal terms ratio is governed by the same parameter Kn as the ratio of higher order derivative terms to the viscous one. However, as we pass to motions with $v \ll c$ the viscous to ideal terms ratio involves an additional small parameter v/c . In the case of properly defined non-relativistic motions obeying the incompressible Navier-Stokes equations the dimensionless parameter governing the viscous to ideal terms ratio is the inverse of the Reynolds number Re

$$Re \sim vL/\nu, \tag{4.33}$$

where v is the characteristic velocity, L is the characteristic scale and ν is the kinematic viscosity, see eq. (1.13). Thus, one can have a situation, where in some regions of space the Reynolds number is not large and viscosity is important, while the Knudsen number is still small everywhere. As long as the viscosity prevents the generation of smaller scales by the dynamics (as it does at least in known cases), one obtains a global solution of the hydrodynamics with viscous term important but higher order derivatives not. In other words, while viscosity is typically a singular perturbation producing order one effects in the solution [17], the higher order terms are often regular perturbations.

The relevance of the above discussion to the viscous shock solution considered before, is that at small Knudsen number, $v(x) \ll c$ in (4.32). Thus, in the frame moving at the speed of sound one deals with the situation of small v/c and the above picture holds. Let us stress again, however, that this solution has no well-defined non-relativistic limit as the shock propagates at a speed of order c .

Note also, that here and below we use the notion of a local Reynolds (Knudsen) number, which is the Reynolds (Knudsen) number defined via the characteristic values of the scale and velocity in the considered region of space [5].

5 Gravity and fluid dynamics

It has been shown in [3], that the four-dimensional CFT hydrodynamics equations are the same as the equations describing the evolution of long-scale perturbations of the five-dimensional black brane. In fact, the derivation of this result parallels exactly the conventional derivation of hydrodynamics equations from the Boltzmann equation. Let us show this in the simplest case of a non-relativistic monoatomic gas.

The Boltzmann equation for a non-relativistic monoatomic gas reads [18]

$$\partial_t f + \mathbf{V} \nabla_{\mathbf{x}} f = Stf, \quad (5.1)$$

where $f(\mathbf{x}, \mathbf{p}, t)$ is the density function in a single particle phase space, $\mathbf{V} = \mathbf{p}/m$ where m is the particles mass, and Stf is the collision integral, which is quadratic in f . Equation (5.1) has a steady-state solution of a Maxwell distribution form

$$f(\mathbf{x}, \mathbf{p}) = \frac{N}{[2\pi m k_B T]^{3/2}} \exp \left[-\frac{m(\mathbf{V} - \mathbf{v})^2}{2k_B T} \right], \quad (5.2)$$

where N is the mean concentration of particles, T is the temperature and \mathbf{v} is an arbitrary velocity present due to the Galilean invariance of eq. (5.1).

The hydrodynamics equations arise when solving the Boltzmann equation by the method of variation of constants. Consider a solution in the form

$$f(\mathbf{x}, \mathbf{p}, t) = f_0(\mathbf{x}, \mathbf{p}, t) + \delta f(\mathbf{x}, \mathbf{p}, t),$$

$$f_0(\mathbf{x}, \mathbf{p}, t) = \frac{N(\mathbf{x}, t)}{[2\pi m k_B T(\mathbf{x}, t)]^{3/2}} \exp \left[-\frac{m[\mathbf{V} - \mathbf{v}(\mathbf{x}, t)]^2}{2k_B T(\mathbf{x}, t)} \right], \quad (5.3)$$

where $\delta f \ll f_0$. This describes the local thermal equilibrium in the gas. The construction of a solution to the Boltzmann equation where f_0 is the zeroth order approximation, produces a series in the Knudsen number Kn , i.e. the ratio of the mean free path to the scale of spatial variations of the fields. Thus, $\delta f \sim Kn f_0$, and the hydrodynamics equations governing the evolution of $N(\mathbf{x}, t)$, $T(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ result as the conditions of the series constructibility.

Analogous derivation of hydrodynamics from the (relativistic) Boltzmann equation can be performed for a CFT. This becomes possible in the limit of the weak coupling where quasiparticle excitations are well-defined (see e.g. [9]). In contrast, at a strong or order unity coupling the Boltzmann equation does not apply. What this means normally is, that for non weakly coupled fluids the hydrodynamics equations are valid but cannot be derived from a microscopic description. The CFT fluids, however, bring a surprise.

Consider the derivation of the CFT hydrodynamics at strong coupling from the equations of gravity. The analogue of eq. (5.1) is given by the five-dimensional Einstein equations with negative cosmological constant

$$R_{mn} + 4g_{mn} = 0, \quad R = -20. \quad (5.4)$$

These equations have a particular "thermal equilibrium" solution - the boosted black brane

$$ds^2 = -2u_\mu dx^\mu dr + \pi^4 T^4 r^{-2} u_\mu u_\nu dx^\mu dx^\nu + r^2 \eta_{\mu\nu} dx^\mu dx^\nu, \quad (5.5)$$

where the temperature T is constant. The presence of an arbitrary constant vector u^μ in the above solution is due to Lorentz invariance, much like the presence of \mathbf{v} in eq. (5.2) due to Galilean invariance. As in eq. (5.3), one looks for a solution of the Einstein equation by the method of variation of constants using the ansatz

$$\begin{aligned}
 g_{mn} &= (g_0)_{mn} + \delta g_{mn}, \\
 (g_0)_{mn} dy^m dy^n &= -2u_\mu(x^\alpha) dx^\mu dr + \pi^4 T^4(x^\alpha) r^{-2} u_\mu(x^\alpha) u_\nu(x^\alpha) dx^\mu dx^\nu + r^2 \eta_{\mu\nu} dx^\mu dx^\nu,
 \end{aligned}
 \tag{5.6}$$

where $y = (x^\mu, r)$. As in the Boltzmann equation, the condition of constructibility of the series solution produces equations for $u^\mu(x^\alpha)$ and $T(x^\alpha)$. These equations are the hydrodynamics equations for a certain class of CFTs in the limit of infinite coupling [3]. As far as the hydrodynamic equations are concerned the infinite coupling means only a particular value of constants in the relations. For example, for dissipative hydrodynamics considered in the previous sections infinite coupling means taking instead of $F(\lambda)$ its appropriate $\lambda \rightarrow \infty$ limit, $F = 1/\pi$. Note, that at the level of the ideal hydrodynamic equations there is no difference at all between different CFTs - the equations are the same. The resulting series for g_{mn} is the series in the Knudsen number of the boundary CFT hydrodynamics.

Thus equations of gravity play for the hydrodynamic equations of a strongly coupled CFTs much the same role as the Boltzmann equation plays at a weak coupling. It is natural to inquire whether the equations of the CFT hydrodynamics allow for a microscopic derivation also at an arbitrary intermediate coupling (say from the equations of gravity with higher order curvature terms in the action). This important question is beyond the scope of this work and is a subject for the future research.

We conclude from the above that the solutions to the CFT hydrodynamics at an infinite coupling describe not only the large-scale motions of the theory, but also the solutions to gravity corresponding to eq. (5.6). We may now use the general results for the CFT hydrodynamics, obtained in the previous section and valid in particular for the hydrodynamics of the infinitely coupled CFT resulting from eqs. (5.4), in order to gain some insight into gravity dynamics. If we consider initial conditions for gravity, that correspond to the initial conditions for hydrodynamics with $v \sim c$ and $Kn \ll 1$ everywhere, then for some limited amount of time we may use ideal hydrodynamics to determine the metric evolution. The ideal hydrodynamic flow is compressible, and it leads to the formation of discontinuities (and maybe other, yet unknown types of singularities) in a finite time. In particular, for "one-dimensional" initial conditions, one may use the exact solutions obtained in the previous sections. Then, ideal hydrodynamics predicts that in a finite time, regions with large metric curvature will appear. They correspond to large values of gradients of the hydrodynamics fields. Eventually, in these regions higher order derivative terms will become important for the evolution of u^μ and T and (expectedly) stop the growth of the curvature. One expects $Kn \sim 1$ and $\delta g \sim g_0$ in these regions.

5.1 The importance of the solutions to non-relativistic hydrodynamics for the construction of global solutions to gravity

As already explained in the last subsection of section 4 the consideration of non-relativistic solutions with $v \ll c$ allows one to construct global solutions of hydrodynamics with $Kn \ll 1$ everywhere. Such solutions are especially important from the viewpoint of the construction of globally valid approximations to the solutions of gravity. Indeed, for them one has the explicit form $g \approx g_0$ of the approximate solution to the Einstein equations (5.4). This is quite similar to the situation for non-relativistic classical gas where the local thermal equilibrium $f \approx f_0$ holds even though viscosity is important for the evolution of the local parameters of equilibrium T , N and v , cf. [18].

The explicit form of the approximate solution to the Einstein equations resulting from the solution to the non-relativistic incompressible Navier-Stokes equations was summarized in eqs. (1.17)–(1.18).

5.2 Global solution for gravity from a small Knudsen number viscous shock

As an example of a global solution to gravity obtained from hydrodynamics we consider the five-dimensional asymptotically AdS gravity solution that corresponds to the viscous shock. Using the results of the previous section we find that the metric g_{vs} defined by

$$\begin{aligned}
 (g_{vs})_{mn} dy^m dy^n = & \sqrt{6} \left[1 - \frac{\sqrt{3}\delta \tanh(x/L)}{2} \right] dt dr - \sqrt{2} \left[1 - \frac{3\sqrt{3}\delta \tanh(x/L)}{2} \right] dx dr \\
 & + r^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{\pi^4 C_2}{3r^2} \left[1 + 2\sqrt{3}\delta \tanh\left(\frac{x}{L}\right) \right] \\
 & \times \left[\left[1 - \sqrt{3}\delta \tanh\left(\frac{x}{L}\right) \right] \frac{3dt^2}{2} - \sqrt{3} \left[1 - 2\sqrt{3}\delta \tanh\left(\frac{x}{L}\right) \right] dx dt \right. \\
 & \left. + \left[1 - 3\sqrt{3}\delta \tanh\left(\frac{x}{L}\right) \right] \frac{dx^2}{2} \right], \\
 L = & \frac{1}{3^{1/4} \sqrt{2} \pi C_2^{1/4} \delta}, \tag{5.7}
 \end{aligned}$$

is a solution of the Einstein equations to order $\delta^2 \ll 1$ (the constant $C_2 > 0$ is arbitrary), and indeed, an explicit calculation verifies that.

Above we used that at small δ one has $u_0 = -[1 - v^2]^{-1/2} \approx -\sqrt{3/2}[1 - \sqrt{3}\delta \tanh(x/L)/2]$ and $u_x = v[1 - v^2]^{-1/2} \approx 1/\sqrt{2}[1 - 3\sqrt{3}\delta \tanh(x/L)/2]$, where $v(x)$ is given by eq. (4.30). As we saw in the previous section the requirement $Kn \ll 1$ imposes that the total variation of v is small and the above solution describes a near equilibrium state.

The above stationary solution describes a non-equilibrium steady state that generalizes the black-brane solution and corresponds to small deviations from the equilibrium. It gives the solution of the Einstein equations with the boundary condition imposed on the boundary stress tensor [19]

$$T_\nu^\mu = -2 \lim_{r \rightarrow \infty} r^4 (K_\nu^\mu - \delta_\nu^\mu), \tag{5.8}$$

where $K_{\mu\nu}$ is the extrinsic curvature tensor to the surface at fixed r and $T^{\mu\nu}$ is the solution of the hydrodynamics equations. If one used the ideal hydrodynamics approximation, where the shock is discontinuous one would obtain a domain wall solution for gravity with no δ -function sources at the wall. At small Kn , the solution (5.7) allows to resolve the inner structure of the wall. At $Kn \sim 1$, the resolution of the wall structure requires the inclusion of the higher order derivatives in the analysis. However, we stress again that the ideal shock solution provides a correct description of the domain wall solution far from wall at any Kn . Finally, we note that it is straightforward to generalize this domain wall solution of gravity by applying special conformal transformations to the solution.

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